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1 Introduction

The ILC is a forthcoming, planned linear collider that will have mainly electron-positron collisions. This will complement the developments at the Large Hadron Collider (LHC), which has proton-proton collisions. So far, the plans include the first stage of the ILC to stretch to thirty one kilometers. As of right now, the site for construction is yet undetermined, but almost 300 laboratories and universities around the world are working on development of the ILC, with approximately 700 people working on design of the accelerator and 900 on the design of the detector [1].

There are two factors that measure the quality of an accelerator: luminosity and energy. The ILC will have the highest luminosity and energy of any similar collider in the world. The goal energy of the colliding beams is around 500 GeV for the first stage of the project, with an option to expand this to 1 TeV in four to ten years after operation begins. To reach these goals, the electric fields that the electrons and positrons will be accelerated with will be around 30-31 MV/m.

The luminosity factors are expected to be around 500 fb^{-1} for the first stage when the energy is 500 GeV, and around 1000 fb^{-1} when the energy is 1 TeV in the second stage. Luminosity is defined as the particle flux per unit of area per second and has the unit of inverse femtobarns per second, where a femtobarn is equal to $10^{-43}m^2$. Although this may seem to be very large, when it is multiplied by the cross section of a process, generally a small number, it will give the total number of interactions recorded in the experiment per second. When equal, Gaussian beams collide head-on, the luminosity reads

$$L = f_0 \frac{N^2}{4\pi\sigma_x\sigma_y},$$

where f_0 is the collision frequency, N the beam population, and $\sigma_{x,y}$ the beam transverse dimensions.

The particles, electrons and positrons, are sent around two circular rings for a short time before being injected into the actual collider towards one another, where they still speed up further before colliding. The actual collider is a long, straight (thus the ”linear” in ILC), narrow tunnel between the injector rings where the interaction takes place. The collision between the two beams takes place in a conducting, cylindrical beam pipe, with a diameter of about 2.5 cm. At the point where they finally collide, called the Interaction Point, each beam is shaped like a long, super thin, strip of paper. They are a hundred thousand times longer than higher, and a hundred times wider than higher. Some of the beam parameters are listed in Table 1.

Parameter	Value
Beam Energy	$500GeV$ (upgradeable to $1TeV$)
Pipe Radius	$2.5cm$
Bunch Length (σ_z)	$0.03cm$
Bunch Height (σ_y)	$0.0057\mu m$
Bunch Width (σ_x)	$0.639\mu m$
Bunch Population (N)	10^{11}

Table 1: ILC beam parameters at the Interaction Point, from Ref.[2].

The goal of this is to get two of these beams that are traveling towards each other at a velocity equal to $(1 - 0.5 \times 10^{-12})$ times the speed of light,[1] or 99.99999999995% of the speed of light, to collide. Obviously, this is a difficult feat to accomplish, and there are many things that can go wrong. The first thing to consider is that the beams must hit each other. As one can imagine, it is not easy to hit a moving target that is 6 nanometers high from many kilometers away.

Once the problem of getting the beams near each other is solved, the question of interest becomes what happens when the beams collide? When they collide, each particle in first beam is attracted towards the center of the second beam by the attractive electro-magnetic force due to the beams having opposite charges. Each particle experiences a transverse force, and the bending produces synchrotron radiation (radiation from accelerating radially) which has been termed beamstrahlung.

Beamstrahlung can and has been used to make sure that the beams collide properly[3], so that optimal transverse overlap and therefore optimal luminosity can be achieved and maintained. Here, we are interested in the part of the beamstrahlung spectrum with wavelengths $\lambda \geq \sigma_z$, the beam length. These are wavelengths in the far infrared and microwave ranges.

When considering this spectrum region, the possibility exists that the beam is radiating coherently, that is, the whole beam radiates as if it were a

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[1] Using $E = \gamma m_e c^2 = \frac{m_e c^2}{\sqrt{1-(\frac{v}{c})^2}}$ and plugging in the right values for E ($500GeV$,) m_e, and c, one can solve for the beam velocity, v, and get $(1 - 0.5 * 10^{-12})c$.

single particle, with each particle wave interfering constructively with each other particle. The opposite, incoherent radiation, is the norm in accelerator radiation phenomena, which means each particle in the beam is radiating on its own, with a net total wave superposition of zero with the other particles.

Radiation intensity of a single particle depends on the charge of the radiating particle squared, $I \propto q^2$. So, incoherent radiation produces the intensity of a single particle times the number of particles in a beam: $I \propto N(q^2)$, where q is the electron charge. Coherent radiation is boosted by another factor of the number of particles in a beam due to superposition, and gives gives $I \propto (Nq)^2$. The difference is a factor $N \sim 10^{11}$, which makes it a potentially very large effect, with numerous consequences.

Ref.[4] does, in fact, provide an estimate for the presence or lack thereof of a coherent regime:

$$\sigma_z < \lambda < 2R\sqrt{2R/\rho}, \tag{1}$$

where R is the beam pipe diameter (25 mm in our case) and ρ is the radius of curvature of the particle during the beam-beam collision.

The radius of curvature is related to the beam parameters including the beam energy and magnetic field as follows

$$\begin{aligned} p \sim E &= qB\rho, \qquad (2) \\ B &\sim \frac{Nqc}{\sigma_x \sigma_z}. \qquad (3) \end{aligned}$$

In present electron-positron colliders, ρ has a typical value of tens of meters, $\sigma_z \sim R$, and Eq. 1 is never satisfied. At the ILC, using the values in Tab. 1, one obtains a curvature radius of order one meter, and

$$0.3\text{mm} < \lambda < 3\text{mm}. \tag{4}$$

Therefore, at the ILC, and only at the ILC, coherent beamstrahlung, in the indicated region, may exist.

This Thesis is the first step towards calculating coherent beamstrahlung. The intended method is via simulation of the beam-beam interaction, using a standard method of dividing the beam plasma into cells [5], and generating and adding the EM waves produced at each step of the interaction.

These waves are related to the radiated intensity $|A(\omega)|^2$, at a given frequency ω, by the following equations [6]

$$\mathbf{A}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathbf{A}(t) e^{i\omega t} dt, \tag{5}$$

$$\frac{d^2 I}{d\omega d\Omega} = 2|\mathbf{A}(\omega)|^2. \tag{6}$$

$\mathbf{A}(t)$ is the EM potential, and is proportional to the electric field of the beam

$$\mathbf{A}(t) = l\mathbf{E}(t) \;\; \mathbf{A}(\omega) = l\mathbf{E}(\omega),$$

where l is the distance from the point of measurement to the radiating point. Thus, at the core of the problem is a calculation of the electric field of the beam.

The electric field is not calculable from first principles. At the beam pipe, the field goes to zero. The goal of this Thesis is to compute the field of the beam in the presence of the boundaries set by the conducting beam pipe.

With this in mind, we used the method of Greene functions in the presence of potential boundary conditions, a so-called Dirichlet problem, to find the electric potential in the pipe from a single beam. From this, the electric field was derived and numerical estimations of the energy density were made. To start the calculations, we started with a couple goals and approximations in mind. We wanted to be accurate in our results to at least one part in one hundred. Using this, we can assume the beam to be one dimensional because the height and width of the beam are both much, much shorter than the length (Table 1).

Dirac delta functions were used to describe these two dimensions, and we then dealt with a line of charge instead of a volume. We treated the beam as an ultra-relativistic beam, since it is so close to the speed of light anyway.

It was easy to see that this problem had cylindrical symmetry. Seeing this, we naturally switched to cylindrical coordinates. After looking at the beam along the axis of the pipe, we then displaced it by an amount $\vec{r} = x_0\hat{x} + y_0\hat{y} = r_0\hat{r} + r_0\theta_0\hat{\theta}$. Due to this symmetry, it's easily seen that under a rotation, we can keep the beam along the x-axis for simplicity: $\vec{r} = x_0\hat{x} = r_0\hat{r}$.

To solve this problem, Dirichlet boundary conditions had to be applied as the beam pipe is at ground ($\Phi(r = R) = 0$). After solving for the electric potential, a gradient calculation was done to find the electric field.

2 Beam in a cylindrical beam pipe.

It is important to discuss first the general properties of the beam and the beam pipe we seek to model. At the International Linear Collider (ILC), the beam typical dimensions are 300μm along the beam pipe axis, chosen to be the $z-$axis, 5.7nm in the vertical direction, chosen to be the $y-$axis, and 639 nm in the horizontal direction, chosen to be the $x-$axis.

The beam pipe diameter is 2.5 cm. With two out of three dimensions well below 0.1% of the beam pipe diameter, the beam charge distribution is

$$\rho(x, y, z, t) = \delta(x)\delta(y)\lambda(z), \tag{7}$$

where $\lambda(z)$ is the linear charge density of the beam

$$\lambda(z) = \frac{Nq}{\sqrt{2\pi}\sigma_z} e^{-(z-ct)^2/2\sigma_z^2}.$$

In Eq. 7, we have made use of the Dirac delta in the two transverse directions, while keeping the Gaussian beam distribution along the direction of motion. The beam is assumed to move at the speed of light, since its relativistic factor γ is about 10^6, which means that the beam velocity β is equal to $1 - 10^{-12}$.

If the beam is not on the center of the beam pipe it is sufficient to replace $\delta(x)\delta(y)$ with $\delta(x - x_0)\delta(y - y_0)$. The beam current density is also easily obtainable:

$$\mathbf{J}(x,y,z,t) = c\rho(x,y,z,t)\hat{\mathbf{z}}. \tag{8}$$

In the extremely relativistic regime we are considering, the electric field produced by the beam is transverse. From Ref.[6], the transverse and longitudinal (z-component) electric fields are, in cylindrical coordinates,

$$E_z = -\frac{q\gamma\beta ct}{(r^2+(\gamma\beta ct)^2)^{3/2}}, \tag{9}$$

$$E_T = \frac{q\gamma r}{(r^2+(\gamma\beta ct)^2)^{3/2}}. \tag{10}$$

Form these formulas, one can easily derive that

$$(E_z)_{max} = \sqrt{4/27}\frac{q}{r^2}, \tag{11}$$

$$(E_T)_{max} = \frac{q\gamma}{r^2}, \tag{12}$$

$$\Delta t \sim \frac{r}{\gamma\beta c}, \tag{13}$$

where Δt is the full width half maximum (FWHM) duration of the interaction with a probe charge located at a radius r. The length of flight $\Delta z = c\Delta t \sim \frac{r}{\gamma\beta} \sim \frac{r}{\gamma}$ during which a charge affects the electric field at the beam pipe ($r = R = 1.25$ cm) is 12.5 nm in our case, which is small compared to the beam length. Further, the longitudinal field is suppressed by a factor of γ compared to the transverse field, and it is set to zero in this calculation.

Our problem simplifies to finding the transverse electric field of a thin line of charge, in a beam pipe. The time dependence of the field is obtained by replacing z with $(z-ct)$, so the problem becomes essentially an electrostatics problem. The magnetic field is then automatically radial, with $B = E/c$, so that the force on a test particle at any given location is given by $F = 2qE$.

The electrostatic problem is solved in the next Section.

3 Calculation of the electric Potential.

The presence of a grounded, conductive beam pipe of radius R is equivalent to setting the electric potential $\Phi(R) = 0$. Conductivity also implies that the electric field components along the beam pipe surface are also zero, $E_\theta = E_z = 0$.

The solution is found combining methods found in Ref.[6] and Ref.[7]. In Ref.[7], the fields were calculated for a beam circulating in a pill box conducting cavity, but the same Fourier series that they use as a starting point is used here later in this Section.

The starting point is Poisson's equation in cylindrical coordinates obtained from Ref.[6]:

$$\frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{\partial^2\Phi}{\partial z^2} = \frac{-\rho}{\epsilon_0}. \tag{14}$$

Since the z component of the electric field is reduced from the other components $(\frac{E_z}{E_T})$ by a factor on the order of γ, E_z is negligible, and the following approximations hold:

$$E_z = 0 \tag{15}$$

$$E_z = \frac{\partial\Phi}{\partial z} \tag{16}$$

$$\frac{\partial\Phi}{\partial z} = 0 \tag{17}$$

$$\frac{\partial^2\Phi}{\partial z^2} = 0. \tag{18}$$

The δ function in cylindrical (radial) coordinates is

$$\delta(x-x_0)\delta(y-y_0)=\frac{\delta(r-r_0)\delta(\theta-\theta_0)}{r},$$

and the density becomes

$$\rho=\frac{Nqe^{-(z-ct)^2/2\sigma_z^2}}{\sqrt{2\pi}\sigma_z}\frac{\delta(r-r_0)\delta(\theta-\theta_0)}{r}. \tag{19}$$

Using this information, Eq. 14 becomes

$$\frac{\partial^2\Phi}{\partial r^2}+\frac{1}{r}\frac{\partial\Phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}=\frac{Nqe^{-(z-ct)^2/2\sigma_z^2}}{\sqrt{2\pi}\epsilon_0\sigma_z}\frac{\delta(r-r_0)\delta(\theta-\theta_0)}{r}. \tag{20}$$

The first fraction on the right hand side is just a constant here, and it is set equal to k to make the equations more presentable.

A form of the solution similar to Ref.[7] is now taken

$$\Phi(r,\theta)=k\Sigma_0^\infty\phi_n(r)e^{in(\theta-\theta_0)}. \tag{21}$$

Two comments are in order: first, the dependence on r_0 is not explicit in Eq. 21, although it will be made explicit shortly. Second, as Fig.[1] shows, the solution must necessarily be even in $(\theta-\theta_0)$, so that

$$\Phi(r,\theta)=k\Sigma_0^\infty\phi_n(r)\cos\left(n(\theta-\theta_0)\right). \tag{22}$$

Derivatives are now evaluated.

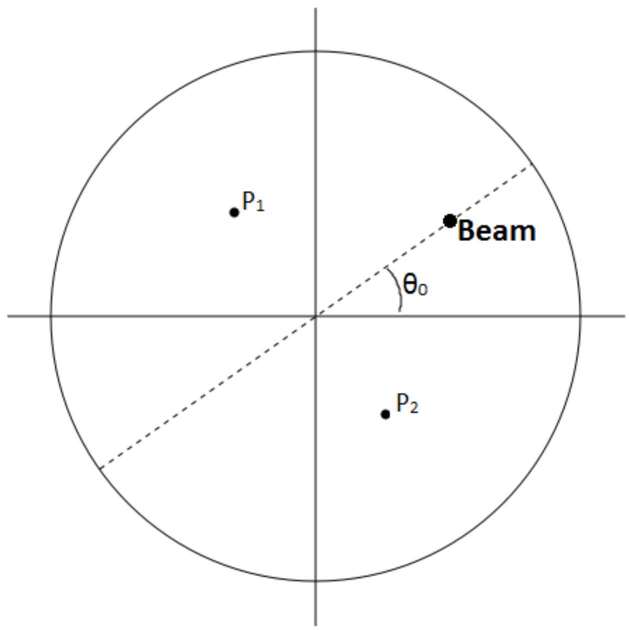

Figure 1: Diagram of the beam pipe with two points, P_1 and P_2 labeled along with θ_0. It is seen in this figure that no matter where the beam is shifted inside of the pipe, the fields created by this beam will be symmetric about the beam; any fields created by a particular beam along this particular θ_0 will be equivalent at P_1 and P_2 due to symmetry.

$$\frac{\partial \Phi}{\partial r} = k\Sigma_0^\infty \frac{\partial \phi_n}{\partial r} \cos\left(n(\theta - \theta_0)\right), \tag{23}$$

$$\frac{\partial^2 \Phi}{\partial r^2} = k\Sigma_0^\infty \frac{\partial^2 \phi_n}{\partial r^2} \cos\left(n(\theta - \theta_0)\right), \tag{24}$$

$$\frac{\partial^2 \Phi}{\partial \theta^2} = -k\Sigma_0^\infty n^2 \phi_n \cos\left(n(\theta - \theta_0)\right). \tag{25}$$

The Poisson equation now becomes, term by term,

$$k\Sigma_0^\infty(\frac{\partial^2\phi_n}{\partial r^2}+\frac{1}{r}\frac{\partial\phi_n}{\partial r}-\frac{n^2}{r^2}\frac{\partial^2\phi_n}{\partial\theta^2})\cos(n(\theta-\theta_0))=\frac{k\delta(r-r_0)\delta(\theta-\theta_0)}{r}. \quad (26)$$

The k factor can be simplified during the solution of the differential equations, and put back at the end when quoting the solution for Φ. Each side of the equation is multiplied by $\cos(m(\theta-\theta_0))$ and integrated over $[0,2\pi]$. The integral values of the l.h.s. are

$$\int_0^{2\pi}\cos(n(\theta-\theta_0))\cos(m(\theta-\theta_0))d\theta=\pi\delta_{nm}$$

if $n\neq 0$ and

$$\int_0^{2\pi}\cos(n(\theta-\theta_0))\cos(m(\theta-\theta_0))d\theta=2\pi\delta_{nm},$$

with δ_{nm} being the Kronecker delta. If $n=0$. The r.h.s integral is equal to 1 for all n,m.

So, for $n=0$ the radial differential equation reads

$$(\frac{\partial^2\phi_0}{\partial r^2}+\frac{1}{r}\frac{\partial\phi_0}{\partial r})=\frac{\delta(r-r_0)}{2\pi r}. \quad (27)$$

and for $n\neq 0$,

$$(\frac{\partial^2\phi_n}{\partial r^2}+\frac{1}{r}\frac{\partial\phi_n}{\partial r}-\frac{n^2}{r^2}\phi_n)=\frac{\delta(r-r_0)}{\pi r}. \quad (28)$$

Eq. 27 can be solved by multiplying by r, and recalling that the Dirac δ is zero everywhere except at r_0. We introduce a dependence on r_0 in the solution

$$\phi_0 = A \ln r + B, \quad r < r_0 \tag{29}$$

$$\phi_0 = C \ln r + D, \quad r > r_0. \tag{30}$$

To avoid infinities at the origin, $A = 0$. The condition $\Phi(R) = 0$ implies that $D = -C \ln R$. Finally, the equality of the two functions at r_0 implies that $B = C \ln r_0/R$.

Having only one unknown C in our equation, we resort to the gradient discontinuity technique in Ref.[6]. We start by multiplying the main differential equation Eq. 27 by r on both sides to get

$$r\frac{\partial^2 \phi_0}{\partial r^2} + \frac{\partial \phi_0}{\partial r} = \frac{\delta(r - r_0)}{2\pi}. \tag{31}$$

We then integrate from r_0^- to r_0^+ to get rid of the delta function:

$$\int_{r_0^-}^{r_0^+} (r\frac{\partial^2 \phi_0}{\partial r^2}) + \int_{r_0^-}^{r_0^+} (\frac{\partial \phi_0}{\partial r}) = \frac{1}{2\pi}. \tag{32}$$

Integrating by parts on the first term on the right, and canceling the resulting integral with the remaining one, we are left with

$$r\frac{d\phi_0}{dr}|_{r_0^+} - \frac{d\phi_0}{dr}|_{r_0^-} = \frac{1}{2\pi} \tag{33}$$

$$C = \frac{1}{2\pi}. \tag{34}$$

The result is $C = 1/2\pi$, so that

$$\begin{aligned} \phi_0 &= \frac{1}{2\pi}\ln r_0/R \;\; r < r_0 && (35) \\ \phi_0 &= \frac{1}{2\pi}\ln r/R, \;\; r > r_0. && (36) \end{aligned}$$

The differential equations for $n \neq 0$ have an extra term, which rules out the logarithmic solution and allows a polynomial solution. It is easy to prove that the solution is

$$\begin{aligned} \phi_n &= Ar^n + Br^{-n}, \;\; r < r_0 && (37) \\ \phi_n &= Cr^n + Dr^{-n}, \;\; r > r_0. && (38) \end{aligned}$$

To avoid infinities at the origin, $B = 0$. The condition $\Phi(R) = 0$ implies that $D = -CR^{2n}$. Finally, the equality of the two functions at r_0 implies that $A = C(1 - (R/r_0)^{2n})$.

Having only one unknown C in our equation, we resort to the gradient discontinuity technique in Ref.[6], as seen used for the case with $n = 0$. We start with Eq. 28, and arrive at

$$\begin{aligned} \frac{d\phi_n}{dr}|_{r_0^+} - \frac{d\phi_n}{dr}|_{r_0^-} &= \frac{1}{\pi r_0} && (39) \\ nCr_0^{n-1} - nDr_0^{-n-1} - nAr_0^{n-1} &= \frac{1}{\pi r_0} && (40) \end{aligned}$$

$$
\begin{aligned}
Cr_0^{n-1} + CR^{2n}r_0^{-n-1} - Cr_0^{n-1}(1-(R/r_0)^{2n} &= \frac{1}{n\pi r_0} && (41)\\
C(r_0^n + R^{2n}r_0^{-n} - r_0^n(1-(R/r_0)^{2n}) &= \frac{1}{n\pi} && (42)\\
C(r_0^n + R^{2n}r_0^{-n} - r_0^n + r_0^n(R/r_0)^{2n}) &= \frac{1}{n\pi} && (43)\\
C(2r_0^n(R/r_0)^{2n}) &= \frac{1}{n\pi} && (44)\\
C(2(R^2/r_0)^n) &= \frac{1}{n\pi} && (45)\\
C &= (1/2\pi n)(r_0/R^2)^n && (46)
\end{aligned}
$$

The result is $C = (1/2\pi n)(r_0/R^2)^n$, so that

$$
\begin{aligned}
\phi_n &= \frac{1}{2\pi n}(r_0 r/R^2)^n(1-(R/r_0)^{2n}) \;\; r < r_0 && (47)\\
\phi_n &= \frac{1}{2\pi n}(r_0 r/R^2)^n(1-(R/r)^{2n}) \;\; r > r_0. && (48)
\end{aligned}
$$

We now reintroduce the quantity k to provide the solutions

$$
\begin{aligned}
\Phi &= \frac{k}{2\pi}(\ln r_0/R + \Sigma_1^\infty (1/n)(r_0 r/R^2)^n(1-(R/r_0)^{2n})\cos(n(\theta-\theta_0)) \;\; r < r_0\\
\Phi &= \frac{k}{2\pi}(\ln r/R + \Sigma_1^\infty (1/n)(r_0 r/R^2)^n(1-(R/r)^{2n})\cos(n(\theta-\theta_0)) \;\; r > r_0
\end{aligned}
$$

According to Ref.[8],

$$\Sigma_1^\infty (a^n/n)\cos(nb) = -\frac{1}{2}\ln(1-2a\cos b + a^2).$$

Substituting, one arrives at the same solution, whether r is greater or lower than r_0

$$\Phi(r,\theta) = \frac{Nqe^{-(z-ct)^2/2\sigma_z^2}}{(2\pi)^{3/2}\epsilon_0\sigma_z} \ln \frac{R^4 + (r_0 r)^2 - 2r_0 r R^2 \cos(\theta - \theta_0)}{R^2(r_0^2 + r^2 - 2r_0 r \cos(\theta - \theta_0))}. \tag{49}$$

Or, more simply,

$$\Phi(r,\theta) = \frac{Nqe^{-(z-ct)^2/2\sigma_z^2}}{(2\pi)^{3/2}\epsilon_0\sigma_z} \ln \frac{R^2 + (r_0/R)^2 r^2 - 2r_0 r \cos(\theta - \theta_0)}{(r_0^2 + r^2 - 2r_0 r \cos(\theta - \theta_0))}. \tag{50}$$

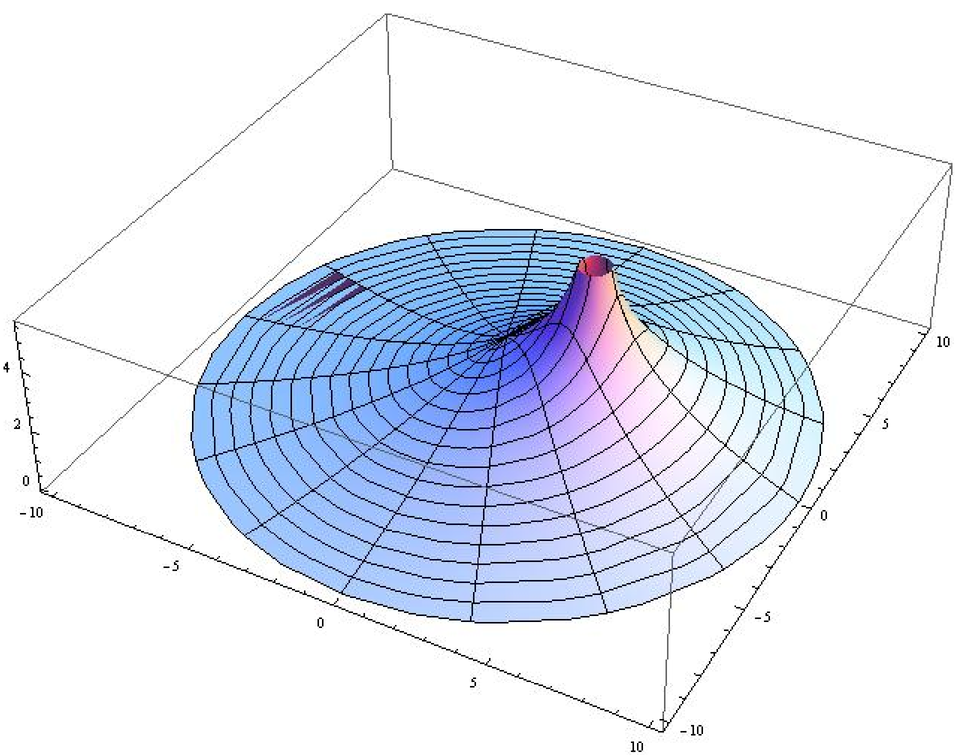

Figure 2: Plot of the Electric Potential with $R = 10$, $r_0 = 3$, $\theta_0 = 0$, and $k = 1$ (Rotated to show details) Ref.[9]

4 Calculation of the Electric Field

Now, we are interested in finding the electric field inside of the pipe that is given by this potential. The equation for finding the electric field from a potential is given by

$$\vec{E} = -\vec{\nabla}\Phi \tag{51}$$

$$\vec{E} = -(\frac{d}{dr}\Phi(r,\theta)\hat{r} + \frac{1}{r}\frac{d}{d\theta}\Phi(r,\theta)\hat{\theta} + \frac{d}{dz}\Phi(r,\theta)\hat{z}) \tag{52}$$

But $\frac{d}{dz}\Phi(r,\theta) = 0$ so we are only left with the r and θ components, which are are simply single derivatives, and turn out to be

$$E_r = -k\frac{2(r_0^2 - R^2)(r(r_0^2 + R^2) - r_0(R^2 + r^2)C(\Delta\theta))}{(r_0^2 + r^2 - 2rr_0C(\Delta\theta))(R^4 + r_0^2r^2 - 2r_0rR^2C(\Delta\theta))} \tag{53}$$

$$E_\theta = -k\frac{2r_0r(r_0^2 - R^2)(R^2 - r^2)S(\Delta\theta)}{(r_0^2 + r^2 - 2rr_0C(\Delta\theta))(R^4 + r_0^2r^2 - 2r_0rR^2C(\Delta\theta))} \tag{54}$$

$$E_z = 0, \tag{55}$$

where $\cos(\theta - \theta_0)$ and $\sin(\theta - \theta_0)$ are abbreviated to $C(\Delta\theta)$ and $S(\Delta\theta)$, respectively, for space purposes.

For graphical purposes, these were also translated to cartesian equations.

$$\begin{aligned} E_x &= k\left[\frac{R^2 x_0 - x r_0^2}{(R^2 - x x_0 - y y_0)^2 + (x y_0 - y x_0)^2} + \frac{x - x_0}{(x - x_0)^2 + (y - y_0)^2}\right] && (56) \\ E_y &= k\left[\frac{R^2 y_0 - y r_0^2}{(R^2 - x x_0 - y y_0)^2 + (x y_0 - y x_0)^2} + \frac{y - y_0}{(x - x_0)^2 + (y - y_0)^2}\right] && (57) \end{aligned}$$

where $r_0 = x_0^2 + y_0^2$.

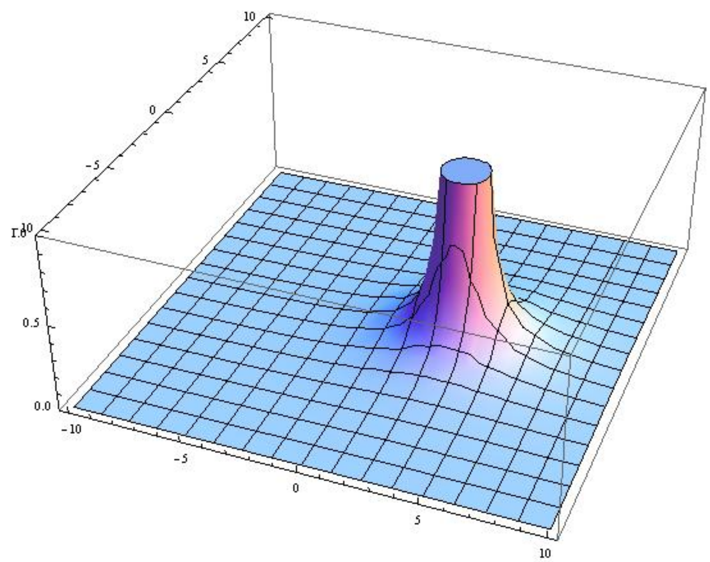

Figure 3: Plot of the Electric Field Squared ($E^2 = E_x^2 + E_y^2$) with $R = 10$, $r_0 = x_0 = 3$, y_0, $\theta_0 = 0$, and $k = 1$ (Rotated to show details) Ref.[9]

As for the electric field at the pipe itself, when $r = R$, we find that $\vec{E_\theta} = \vec{E_z} = 0$, as we would expect.

Now we need a numerical computation of $\int E^2 da$, which, in radial coordinates, becomes $\int E^2 r dr d\theta$. We started with the assumption that $r_0 = \theta_0 = 0$ to make sure when we did calculate this for an off center beam, we were on the right track.

When $r_0 = \theta_0 = 0$

$$\Phi(r,\theta) = \frac{Nqe^{-(z-ct)^2/2\sigma_z^2}}{(2\pi)^{3/2}\epsilon_0\sigma_z}\ln\frac{R^2}{r^2}$$

Or simply

$$\Phi(r,\theta) = k\ln\frac{R^2}{r^2},$$

Since $\vec{E} = -\nabla\Phi$, $\vec{E_r} = 2k/r$, $\vec{E_\theta} = \vec{E_z} = 0$, we have $E^2 = 4k^2/r^2$.

Let $S = \int\int E^2 da$ be a two-dimensional representation of energy density, and that to find the actual energy density, we would integrate this expression in the z direction.

$$\begin{aligned} S_0 &= \int\int E^2 da && (58)\\ &= \int_\epsilon^R \int_0^{2\pi} \frac{4k^2}{r^2} r dr d\theta && (59)\\ &= \int_\epsilon^R \int_0^{2\pi} \frac{4k^2}{r} dr d\theta && (60)\\ &= 8\pi k^2 \ln\left(\frac{R}{\epsilon}\right) && (61)\\ & && (62)\end{aligned}$$

Where ϵ is used instead of zero due to an infinite limit that will be set to a small radius around the beam, and S_0 is just the two-dimensional energy density of a beam centered on the axis. S_0 is named because we can treat the next section as a perturbation of this one, describing the final result in terms of S_0 and correction terms.

If we assume that the beam is off center by a small amount on the x-axis, $\delta = x_0/R = r_0/R$, then we have $\theta_0 = 0$. Rewriting the potential with this in mind, we find that the potential becomes

$$\Phi = k \ln \frac{\delta^2 r^2 - 2R\delta r \cos\theta + R^2}{r^2 - 2R^2\delta cos\theta + R^2\delta^2} \tag{63}$$

When δ is small, we can make the assumption that $-1 < \delta < 1$ and we can express this potential in a Taylor Series expansion in δ as follows

$$\Phi = k[\ln \frac{R^2}{r^2} + (2\cos(\theta)(\frac{R}{r} - \frac{r}{R})\delta + (1 - 2\cos(\theta)^2)(\frac{1}{r^2R^2})(r^4 - R^4)\delta^2 + O(\delta^3)] \tag{64}$$

Higher order terms that include the factor $\delta^n | n \geq 3$ can be ignored because they only add to this approximation by a very minute amount, and the accuracy that we have is quite good enough for the experiment at hand.

Calculating the components of the electric field in this simplified approximation is much easier to do, since we are now dealing with derivatives of a quadratic polynomial as opposed to declivities of a logarithm involving polynomials. As with before,

$$\vec{E} = -\vec{\nabla}\Phi \tag{65}$$

$$\vec{E} = -(\frac{d}{dr}\Phi(r,\theta)\hat{r} + \frac{1}{r}\frac{d}{d\theta}\Phi(r,\theta)\hat{\theta} + \frac{d}{dz}\Phi(r,\theta)\hat{z}) \tag{66}$$

$$\frac{d}{dz}\Phi(r,\theta) = 0 \quad (67)$$

The radial and angular components are also calculated the same as before, simple single derivatives of the potential.

$$E_r = \frac{d}{dr}(k(\ln\frac{R^2}{r^2} + (2\cos(\theta)(\frac{R}{r} - \frac{r}{R})\delta + (1-2\cos(\theta)^2)(\frac{(r^4-R^4)}{r^2R^2})\delta^2)) \quad (68)$$

$$E_r = k(-\frac{2}{r} - \frac{2\delta\cos(\theta)}{r^2R}(R^2-r^2) + \frac{2\delta^2}{r^3R^2}(r^4+R^4)(1-2\cos(\theta)^2)) \quad (69)$$

$$E_\theta = \frac{1}{r}\frac{d}{d\theta}(k(\ln\frac{R^2}{r^2} + (2\cos(\theta)(\frac{R}{r} - \frac{r}{R})\delta + (1-2\cos(\theta)^2)(\frac{(r^4-R^4)}{r^2R^2})\delta^2)) \quad (70)$$

$$E_\theta = k(-\frac{2\delta}{r^2R}(R^2-r^2)\sin(\theta) + \frac{4\delta^2}{r^3R^2}(r^4-R^4)\cos(\theta)\sin(\theta)) \quad (71)$$

Now we again need a numerical computation of $\int E^2 da$, which, in radial coordinates, becomes $\int E^2 r dr d\theta$. Neglecting $O(\delta^n), n \geq 3$ terms, we're left with

$$\begin{aligned} S &= \int_\epsilon^R \int_0^{2\pi} r(E_r^2 + E_\theta^2) dr d\theta && (72) \\ &= \int_\epsilon^R 8\pi k^2 r(\frac{R^4r^4 + r^2(R^3 - Rr^2)^2\delta^2}{R^4r^6})dr && (73) \\ &= 4\pi k^2((\frac{r^2}{R^2} - \frac{R^2}{r^2})\delta^2 + (2-4\delta^2)\ln r)|_{r=\epsilon}^{r=R} && (74) \\ &= 8\pi k^2 \ln(\frac{R}{\epsilon}) - 4\pi k^2\delta^2(4\ln(\frac{R}{\epsilon}) + (\frac{R^4+\epsilon^4}{R^2\epsilon^2})) && (75) \\ &= S_0 - \delta^2(\frac{S_0}{2} + 4\pi k^2(\frac{R^4+\epsilon^4}{R^2\epsilon^2})) && (76) \end{aligned}$$

Where ϵ is a lower boundary close to the source. Since we used a delta function for the source, at the charge, any electric field or energy density would be infinite without this bound that is not zero. This is as was expected: the initial two-dimensional energy density with correction terms of δ^2.

5 On the Possibility of Beamstrahllung

In the following equations, the variables are as follows: E is the beam energy, p is the momentum of the bunch, $\sigma_x, \sigma_y, \sigma_z$ are the bunch length, height (which is negligible for these calculations) and length, R is the radius of the beam pipe, B is the magnetic field immediately around the beam, N is the number of particles in a bunch, and ρ is the radius of curvature for the trajectory of the beam.

As seen in Ref.[4], to have coherent synchrotron radiation, the following condition must be met:

$$2R\sqrt{\frac{2R}{\rho}} > \lambda > \sigma_z \tag{77}$$

We already have the beam pipe radius and the length of the bunch, so we now need to find the radius of curvature of the beam. We assume that the two beams are very close to eachother and the distance between them, h, is mostly negligible. We then look at the motion of the first beam in the presence of the magnetic field of the second. Balancing forces, and assuming the field will be perpendicular to the beam, we have

$$F_{magnetic} = F_{centripetal} \tag{78}$$

$$evB = \frac{mv^2}{\rho} \tag{79}$$

solving for ρ, we get

$$\rho = \frac{p}{eB}, \tag{80}$$

where p is the momentum of the beam.

From elementary special relativity,

$$E^2 = p^2c^2 + (m_e)^2c^4 \tag{81}$$

$$p = \sqrt{\frac{E^2}{c^2} - (m_e)^2c^2} \tag{82}$$

or since $\frac{E^2}{c^2} \gg (m_e)^2c^2$,

$$p = \frac{E}{c} \tag{83}$$

Now, we find the magnetic field due to the second beam at a height, h, above the beam. Since this height is very small, and also the beam is very thin, we can treat the beam as an infinitely large, charged plane, moving under the point of measurement at a velocity c. This was set up such the beam was moving in the $\hat{z}$ direction, spanned the xz axis, and the observation point was at $h\hat{y}$. From Ref.[10], we have

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{K}(\vec{r'}) \times \vec{\mathbf{r}}}{\mathbf{r}^2} da \tag{84}$$

where $\vec{K}$ is the surface current density, $\vec{B}$ is the magnetic field, $\vec{r}$ is the postion of measurement, $\vec{\mathbf{r}}$ is the position of charge relative to the point of observation, and da' is the differential area of the surface of charge. From the statement of the problem, we have the surface current density (K) equal to the surface charge density (σ) times the beam velocity (v)

$$\vec{K} = \sigma\vec{v} \tag{85}$$

$$\vec{K} = -\frac{Nec}{\sigma_z\sigma_x}\hat{z} \tag{86}$$

$$\vec{\mathbf{r}} = \frac{x\hat{x} + h\hat{y} + z\hat{z}}{\sqrt{x^2 + h^2 + z^2}} \tag{87}$$

$$\mathbf{r}^2 = x^2 + h^2 + z^2 \tag{88}$$

$$da' = dxdz \tag{89}$$

Putting these variables into the magnetic field, we have

$$\vec{B} = -\frac{\mu_0 Nec}{4\pi\sigma_z\sigma_x}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{x\hat{y} - h\hat{x}}{(x^2 + h^2 + z^2)^{\frac{3}{2}}}dxdz \tag{90}$$

$$\vec{B} = \frac{\mu_0 Nec}{2\sigma_z\sigma_x}\hat{x} \tag{91}$$

After combining the previous results, we find that

$$\rho = \frac{2\sigma_z\sigma_x E}{c^2 e^2 \mu_0 N} \tag{92}$$

We then looked at this radius of curvature and the condition for CSR to see if it was met for the two accelerators in question. Below is a short table of different properties from each accelerator:

Variable [4]	ILC	CESR
N	$2 * 10^{10}$	$1.15 * 10^{10}$
σ_x	$6.39 * 10^{-7} m$	$4.60 * 10^{-4} m$
σ_z	$3 * 10^{-4} m$	$1.8 * 10^{-2} m$
$2R$	$12.5 * 10^{-3} m$	$12.5 * 10^{-3} m$
E	$500 GeV = 8.01 * 10^{-8} J$	$E = 6 GeV = 9.61 * 10^{-10} J$

Table 2: Specific beam parameters for ILC and CESR

Using the variables given for the ILC, $\rho = 0.530m$, so

$$2R\sqrt{\frac{2R}{\rho}} > \lambda > \sigma_z \tag{93}$$

$$5.4 * 10^{-3} m > \lambda > 3 * 10^{-4} m \tag{94}$$

and the condition for CSR is met.

On the other hand, using the given variables for CESR, $\rho = 274m$, and

$$2R\sqrt{\frac{2R}{\rho}} > \lambda > \sigma_z \tag{95}$$

$$7.6 * 10^{-3} m \not> \lambda > 1.8 * 10^{-2} m. \tag{96}$$

The condition for CSR in a typical modern e^+e^- linear accelerator, CESR, is not met. Similar calculations can be done for other accelerators, and it will be found that the ILC will be the first accelerator that will create the right conditions for us to be able to observe, and possibly harness, CSR.

6 Conclusions

Through this thesis, calculations were done to find the electric field of a narrow beam of charge in a long, conducting, hollow cylinder, and also to comment on the possibility of observing coherent beamstrahllung at the International Linear Collider.

Using simple approximations and parts of different techniques ([4], [6] [10]) we were able to calculate the electric field of the electron/positron beams at the ILC to approximately 1% error. This is a useful starting point for any other calculations to be made in the future that are more rigorous. From the electric field calculations, we derived an expression for a type of energy density.

Using the electric field again, along with given information about the collider itself, we were able to use an estimate of the parameters needed to observe coherent beamstrahllung. As it turns out, at the ILC and no other collider in the world, we will have the opportunity for the first time to observe and study this phenomenon. Along with just the observation of coherent beamstrahllung, it's practical uses are numerous.

The ILC has the potential to create millimeter wavelength lasers with higher energy than ever before. As with any type of laser, it may have many uses. One use would be in the field of millimeter wavelength chemistry, where a laser like this would be in high demand.

When the detectors at the ILC will record a reaction, it will be compared to theory to find correlations. If coherent beamstrahllung is observed, the signals will be approximately up to $10^{1}1$ as high as just normal beamstrahllung due to the constructive interference. If this is not taken into consideration, and the detectors would be built for energies many orders of magnitudes lower than this thesis predicts, then the detectors could possibly be severly damaged by the energies released. There is also a factor of cooling and shielding the detector. Again, if precautions are not taken for the ideas raised here, even in the accelerating chambers, the particles can have destructive energies because of the shear magnitude difference between the energies of coherent beams and non-coherent beams.

7 References

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